Probability and Random Processes
ECS 315

Asst. Prof. Dr. Prapun Suksompong

prapun@siit.tu.ac.th

6.1 Conditional Probability

Office Hours:
BKD 3601-7

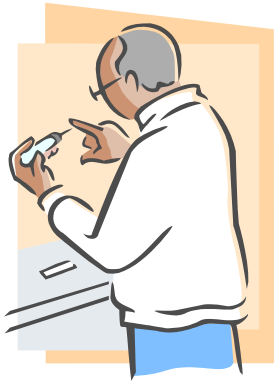
Monday	**14:00-16:00**
Wednesday	**14:40-16:00**

Suppose we have a diagnostic test for a particular disease which is 99% accurate. The test gives a positive result.

What is the probability that the person actually has the disease?

Disease Testing

- Suppose we have a diagnostic test for a particular **disease** which is 99% accurate.
- A person is picked at random and tested for the disease.
- The test gives a **positive result**.
- Q1: What is the probability that the person actually has the disease?
- Natural answer: 99% because the test gets it right 99% of the times.



99% accurate test?

- Two kinds of error
- If you use this test on many persons **with** the disease, the test will indicate correctly that those persons have disease 99% of the time.
  - **False negative** rate = 1% = 0.01 1 → 0
- If you use this test on many persons **without** the disease, the test will indicate correctly that those persons do not have disease 99% of the time.
  - **False positive** rate = 1% = 0.01 0 → 1

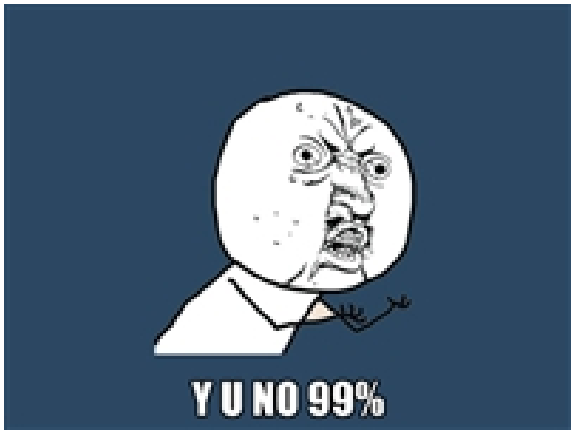
Disease Testing: The Question

- Suppose we have a diagnostic test for a particular **disease** which is 99% accurate.
- A person is picked at random and tested for the disease.
- The test gives a **positive result**.
- Q1: What is the probability that the person actually has the disease?
- Natural answer: 99% because the test gets it right 99% of the times.
- Q2: Can the answer be 1% or 2%?
- Q3: Can the answer be 50%?

Disease Testing: The Answer

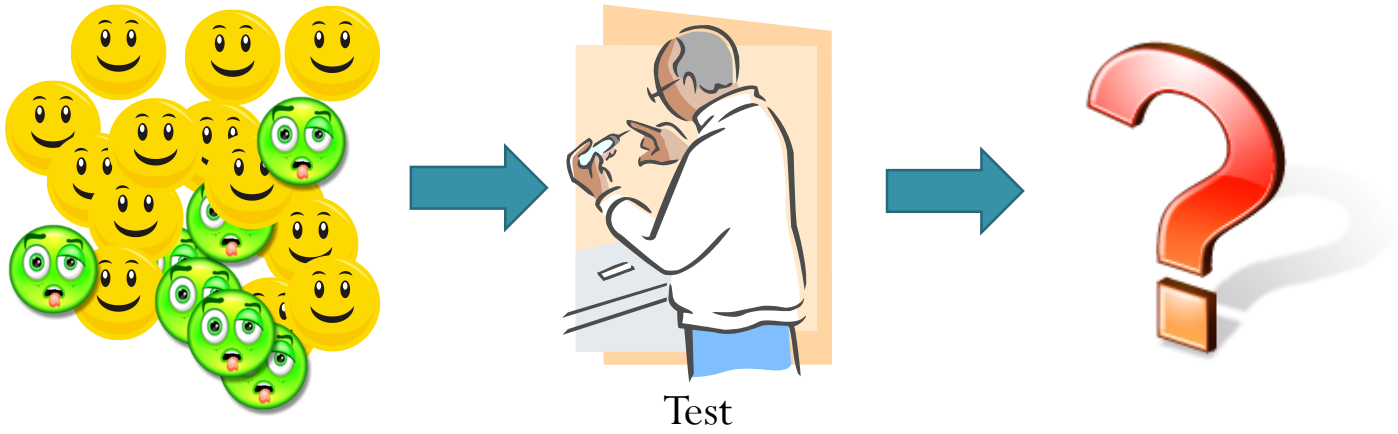
Q1: What is the probability that the person actually has the disease?

A1: The answer actually depends on how **common** or how **rare** the disease is!



Why?

- Let's assume **rare disease**.
  - The disease affects about 1 person in 10,000.
- Try an experiment with **10^6 people**.
- Approximately **100 people** will have the disease.
- What would the (99%-accurate) test say?



10^6 people

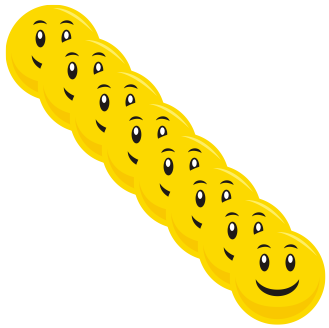
Results of the test

100 people w/ disease

approximately

99 of them will test positive

1 of them will test negative



999,900 people w/o disease

989,901 of them will test negative

9,999 of them will test positive

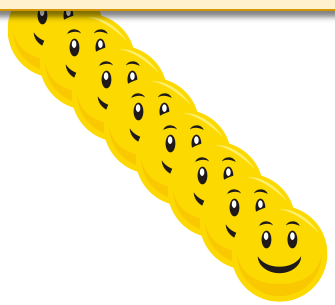
Results of the test

100 people w/ disease

99 of them will test positive

1 of them will test negative

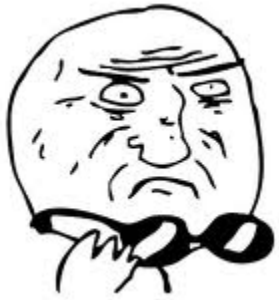
Of those who test positive, only $\frac{99}{99+9,999} \approx 1\%$ actually have the disease!



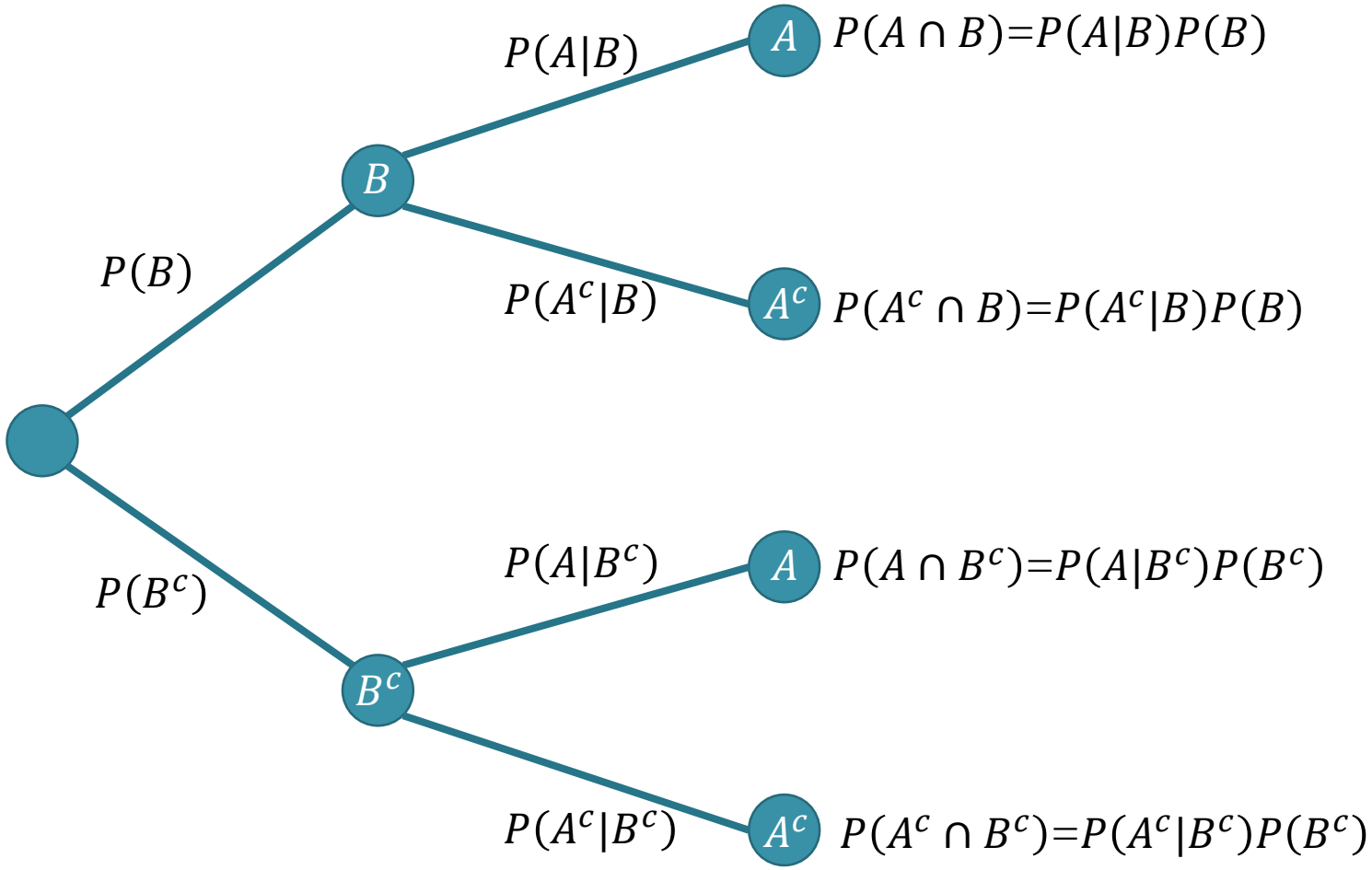
999,900 people w/o disease

989,901 of them will test negative

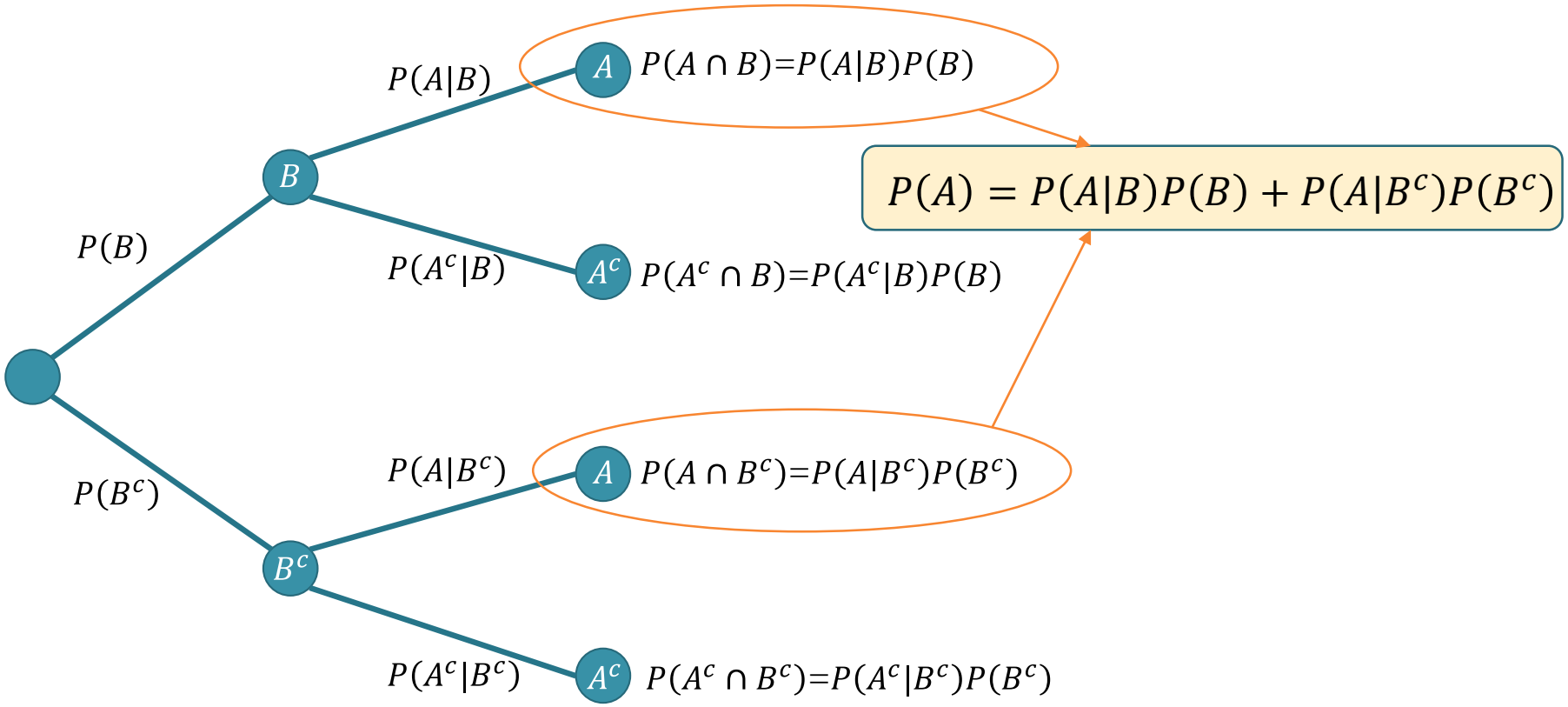
9,999 of them will test positive



Tree Diagram and Conditional Probability

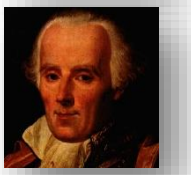


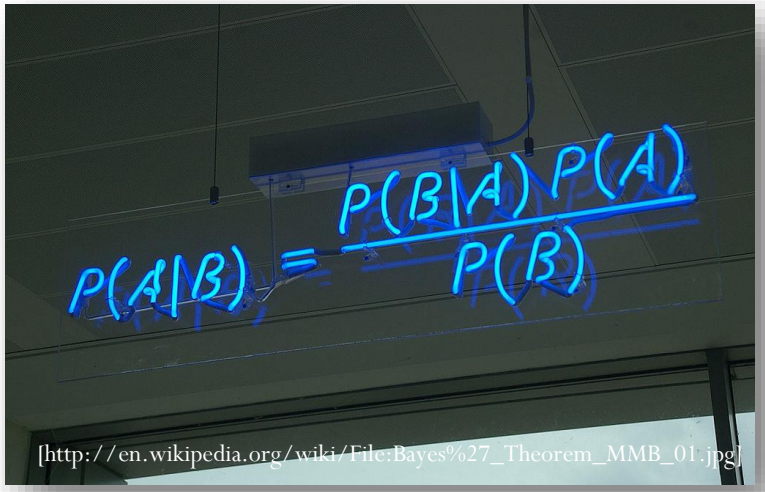
Tree Diagram and Total Probability Theorem



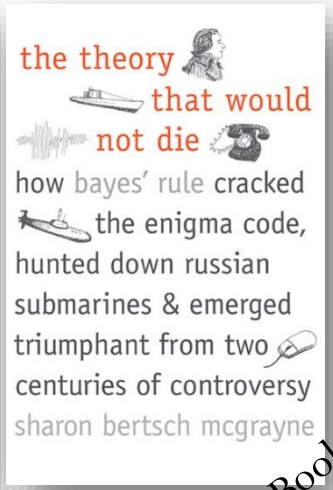
Bayes' Theorem: History

- Named after the **Thomas Bayes** (1701–61)
  - Father of mathematical decision making
- Bayes studied how to compute a distribution for the probability parameter of a binomial distribution in 1740s.
- His friend Richard Price edited and presented this work in 1763, after Bayes's death, as "An Essay towards solving a Problem in the Doctrine of Chances".
- Laplace independently rediscovered and extended Bayes' results in 1774.
  - Over the next forty years he developed it into the form we use today.

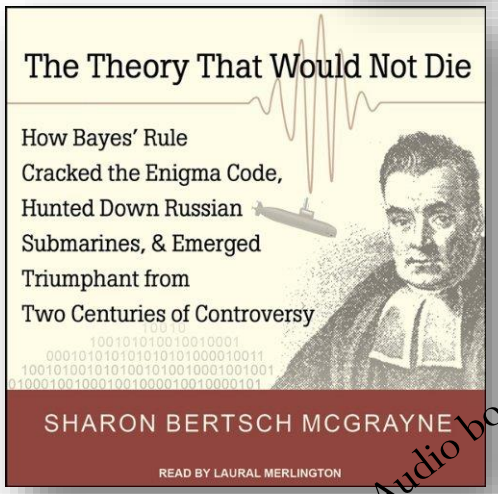




[http://en.wikipedia.org/wiki/File:Bayes%27_Theorem_MMB_01.jpg]



Book



Audio book

Bayes' Theorem: Scientific Battle

- An example of "science gone awry".
- The scientific battle over Bayes' theorem (Bayesian analysis) is lasted for 150 years.
  - Respected statisticians rendered it professionally taboo
    - while practitioners relied on it to solve problems
- Similar case: Geologists accumulated the evidence for Continental Drift in 1912 and then spent 50 years arguing that continents cannot move.
- Sometime during the 1740s, Bayes made this discovery but then mysteriously abandoned it.
  - Bayes' theorem began life amid an inflammatory religious controversy in England in the 1740s: can we make rational conclusions about God based on evidence about the world around us?
- Laplace gave it its modern mathematical form and scientific application and then moved on to other methods.

Bayes' Theorem

Using the concept of **conditional probability** and **Bayes' Theorem**, you can show that

> the probability that a person will have the disease given that the test is positive

is given by

$$\frac{(1-p_{TE})p_D}{(1-p_{TE})p_D + p_{TE}(1-p_D)}$$

where, in our example,

$p_D = 10^{-4}$

$p_{TE} = 1 - 0.99 = 0.01$

Bayes' Theorem

Using the concept of **conditional probability** and **Bayes' Theorem**, you can show that

the probability $P(\mathrm{D}\,|\,\mathrm{T_P})$ that a person will have the disease given that the test result is positive

is given by

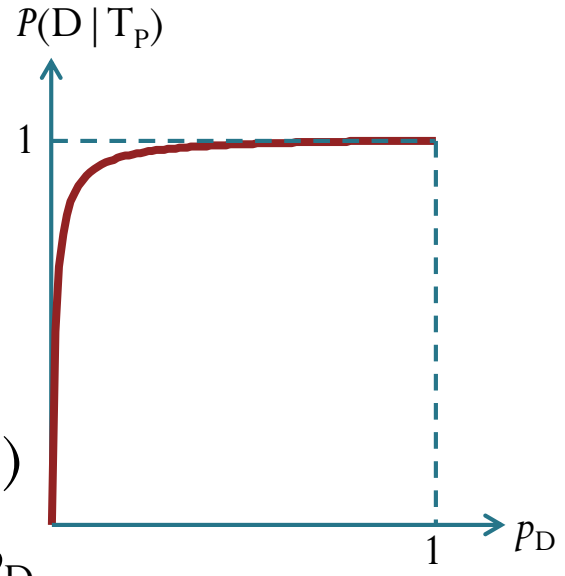
$$\frac{(1-p_{TE})\,p_D}{(1-p_{TE})\,p_D + p_{TE}\,(1-p_D)}$$

When different value of p_D is assumed,

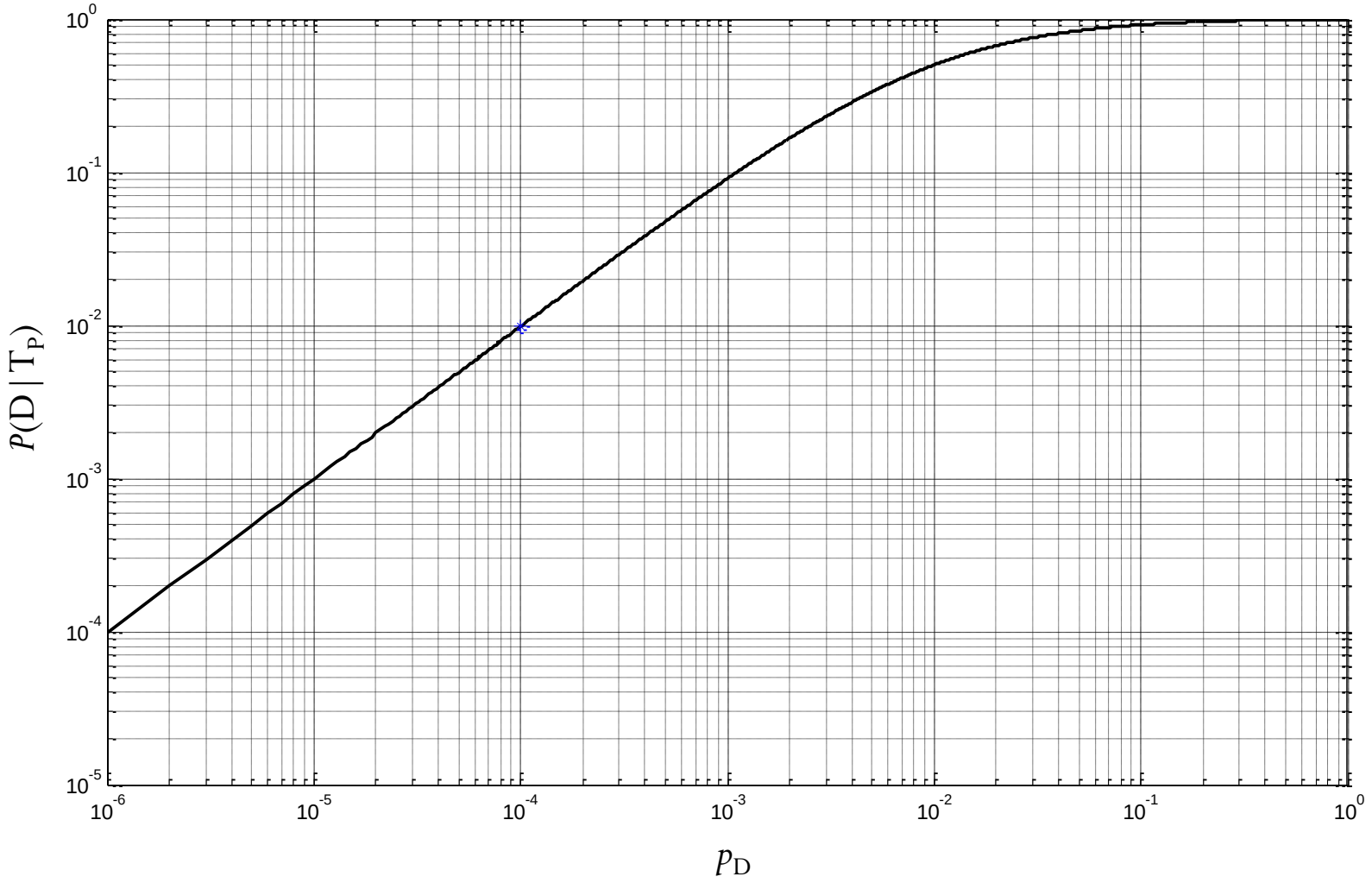
We get different value of $P(\mathrm{D}\,|\,\mathrm{T_P})$.

Conclusion: ***Any*** value (between 0 and 1)

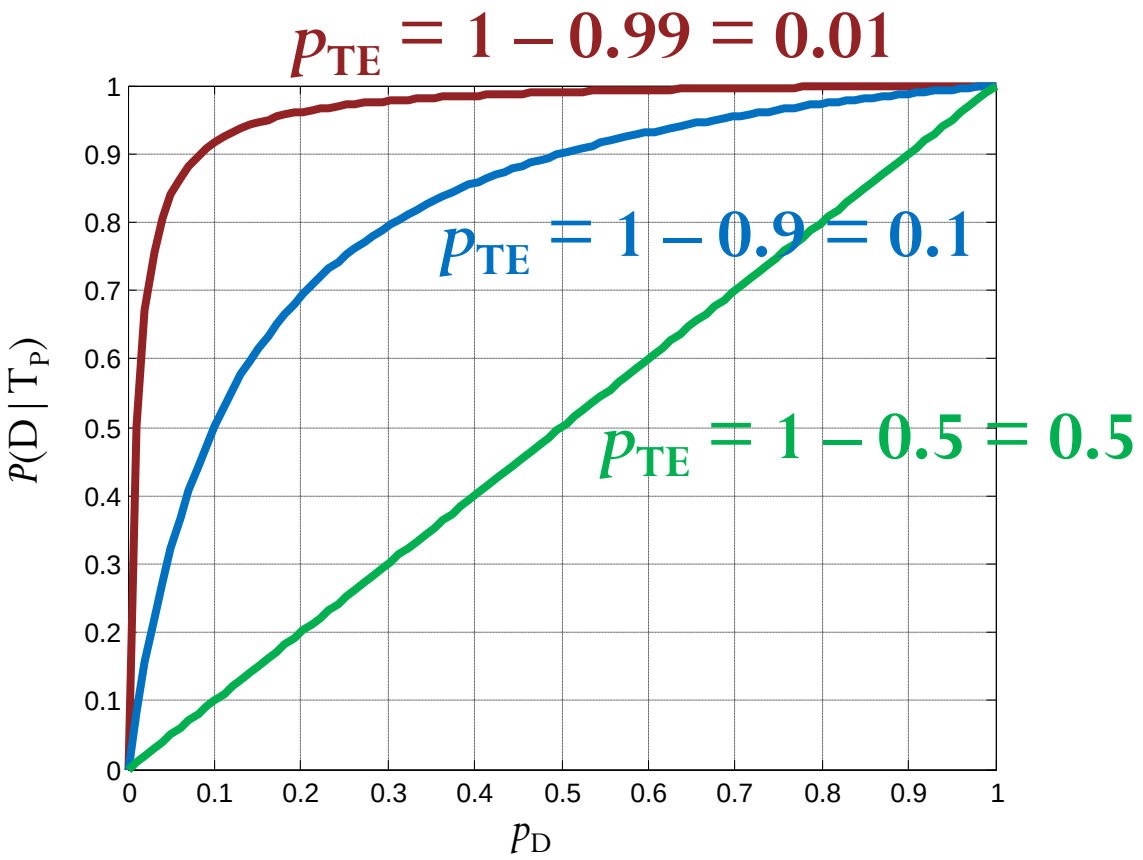
can be obtained by varying the value of p_D



In log scale…


10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
$\mathcal{P}(\mathrm{D}\,|\,\mathrm{T_P})$
10^{-6}
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
p_{D}

Effect of p_{TE}

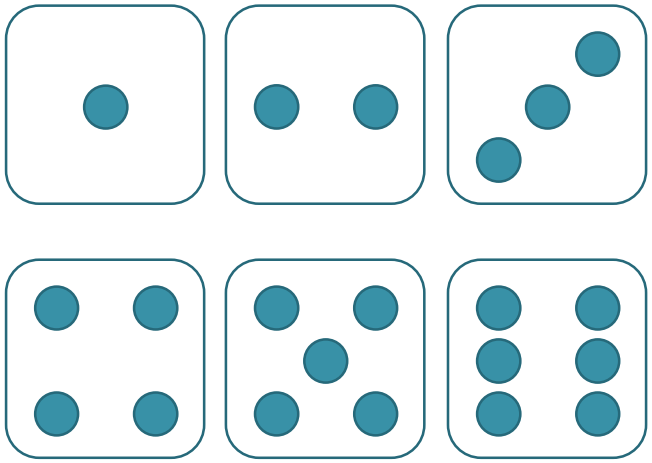


Wrap-up

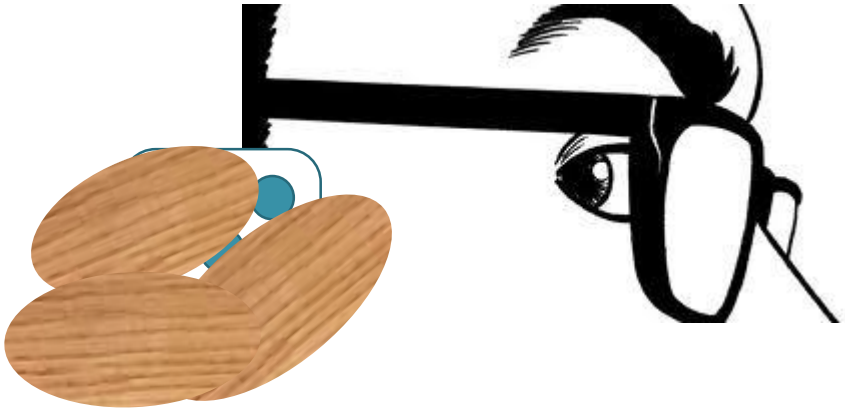
- Q1: What is the probability that the person actually has the disease?
- A1: The answer actually depends on how common or how rare the disease is! (The answer depends on the value of P_D.)
- Q2: Can the answer be 1% or 2%?
- A2: Yes.
- Q3: Can the answer be 50%?
- A3: Yes.

Example: A Revisit

- Roll a fair dice



- Sneak peek:



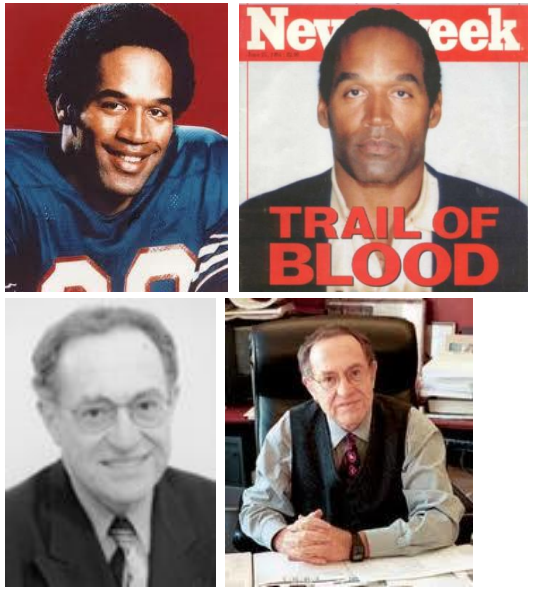
Prosecutor's fallacy

- Criminal trial for murder
  - "one of the biggest media events of 1994–95"
  - "the most publicized criminal trial in American history"
  - Often characterized as "the trial of the century"

(การพิจารณาคดีในศาล)

- O. J. Simpson
  - At the time a well-known celebrity famous both as a **TV actor** and as a retired **professional football star**.
- Defense lawyer: Alan Dershowitz
  - Renowned attorney and **Harvard Law School professor**

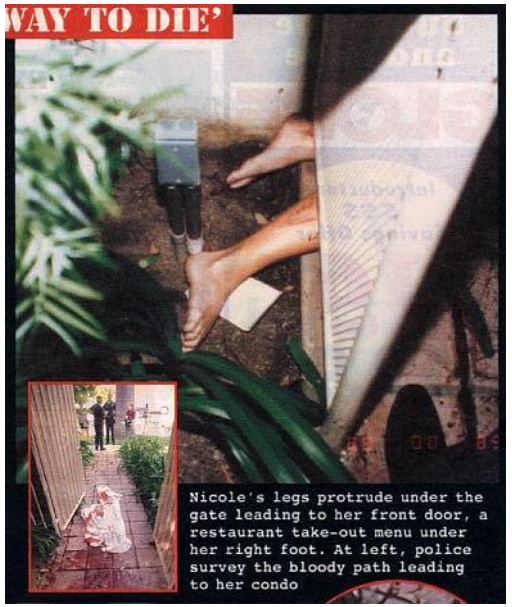
(ทนาย)

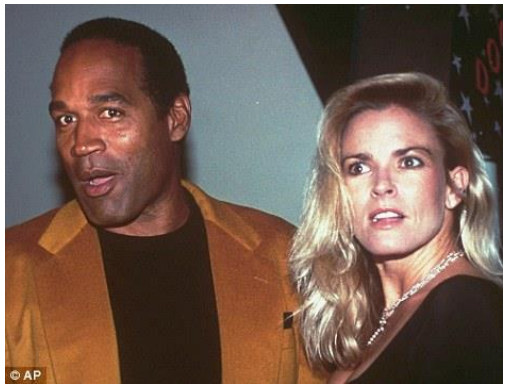


[Mlodinow, 2008, p. 119-121],[Tijms, 1007, Ex 8.7]

The murder of Nicole

- **Nicole** Brown was murdered at her home in Los Angeles on the night of June 12, 1994.
  - So was her friend Ronald Goldman.
- The **prime suspect** (ผู้ต้องสงสัย) was her (ex-) **husband** O.J. Simpson.
  - (They were divorced in 1992.)





Prosecutor* = a government official who conducts criminal prosecutions on behalf of the state
(พนักงานอัยการ) (เป็นฝ่ายผู้ฟ้องร้อง/โจทก์)

Prosecutors' argument

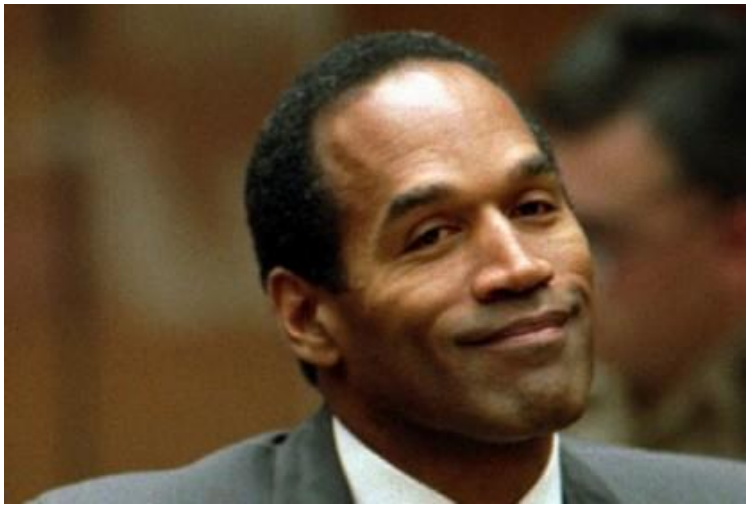
- Prosecutors* spent the first ten days of the trial entering **evidence** of Simpson's history of **physically abusing** her and claimed that this alone was a good reason to suspect him of her murder.
- As they put it,
  "a slap is a prelude to homicide." (ฆาตกรรม)

Counterargument

- The **defense attorneys** (ทนายฝ่ายจำเลย) argued
  - that the prosecution* had spent two weeks trying to **mislead** the jury
  - and that the **evidence** that O. J. had battered (ทุบตี) Nicole on previous occasions **meant nothing**.
- **Dershowitz's reasoning**:
  - 4 million women are battered annually by husbands and boyfriends in the US.
  - In 1992, a total of 1,432, or 1 in 2,500, were killed by their (ex)husbands or boyfriends.
  - Therefore, few men who slap or beat their domestic partners go on to murder them.
- True? …Yes…Convincing?

The verdict:

Not guilty for the two murders!



The verdict was seen live on TV by more than half of the U.S. population, making it one of the most watched events in American TV history.

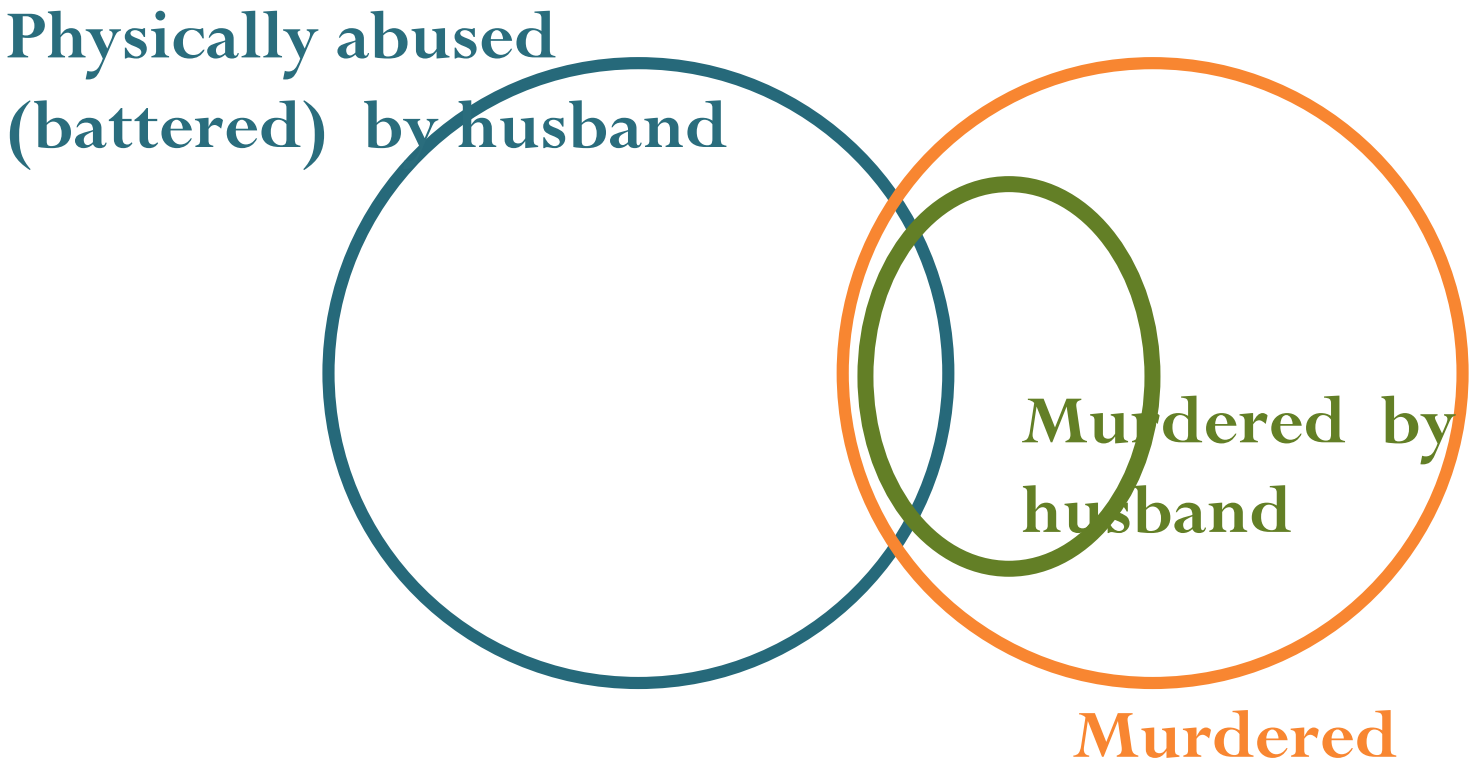
The Truth: Another number...

- It is important to make use of the crucial fact that Nicole Brown was murdered.
- The relevant number is not the probability that a man who batters his wife will go on to kill her (1 in 2,500) but rather the probability that a battered wife who was murdered was murdered by her abuser.

This event has happened and should be used in probability evaluation

- According to the Uniform Crime Reports for the United States and Its Possessions in 1993, the probability Dershowitz (or the prosecution) should have reported was this one: of all the battered women murdered in the United States in 1993, some 90 percent were killed by their abuser.
- That statistic was **not mentioned at the trial**.

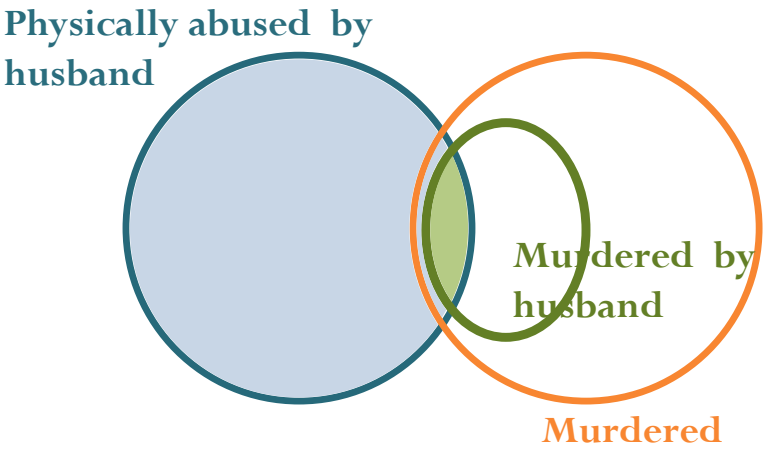
A Simplified Diagram

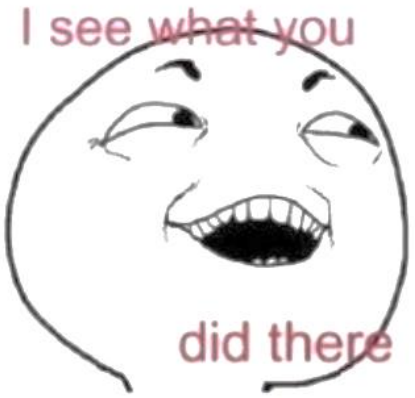


Probability Comparison

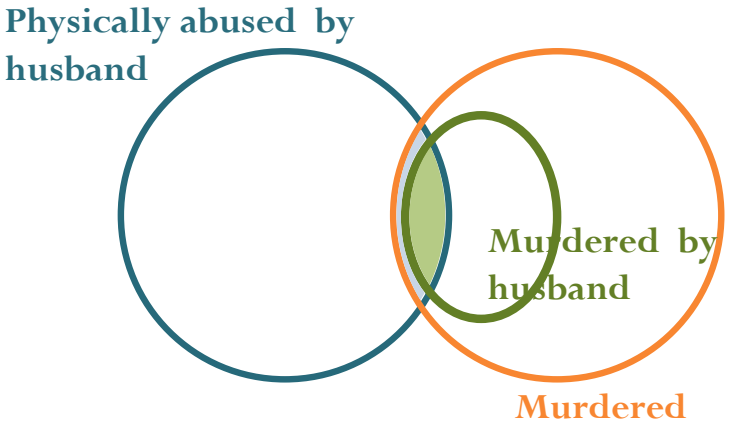
The orange event is ignored.

1 in 2,500
(0.04%)





90%



The Whole Truth …

- Dershowitz may have felt justified in **misleading** the jury because, in his words, "the courtroom oath—'to tell the truth, the ***whole truth*** and nothing but the truth'—is applicable only to witnesses.

- Defense attorneys, prosecutors, and judges don't take this oath . . . indeed, it is fair to say the American justice system is built on a foundation of not telling the whole truth."

[Mlodinow, The Drunkard's Walk: How Randomness Rules Our Lives]

Epilogue

- 1995: O. J. Simpson was ***acquitted*** of the 1994 murder of Nicole Brown Simpson and Ronald Goldman.
  - Polls showed that most African-Americans felt that justice had been served by the "not guilty" verdict, while most white Americans did not.
- In 1997, a civil jury unanimously found Simpson liable for the wrongful death of Ronald Goldman and stabbing of Nicole Brown.
  - Simpson was ordered to pay $33,500,000 in damages.
- In 2007, Simpson was arrested in Las Vegas, Nevada, and charged with numerous felonies, including armed robbery and kidnapping.
- In 2008, he was found guilty and sentenced to 33-years imprisonment, with a minimum of nine years without parole.

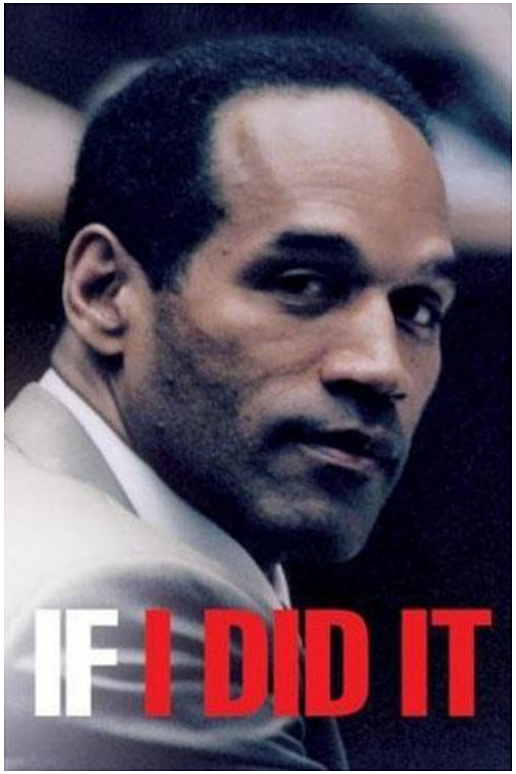
[http://abcnews.go.com/US/oj-simpson-denied-trial-robbery-kidnapping-case/story?id=21025986]

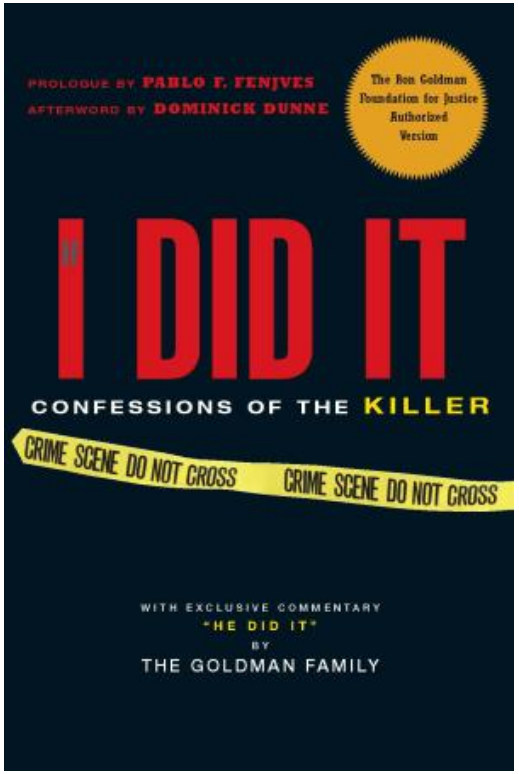
Civil Trial vs. Murder Trial

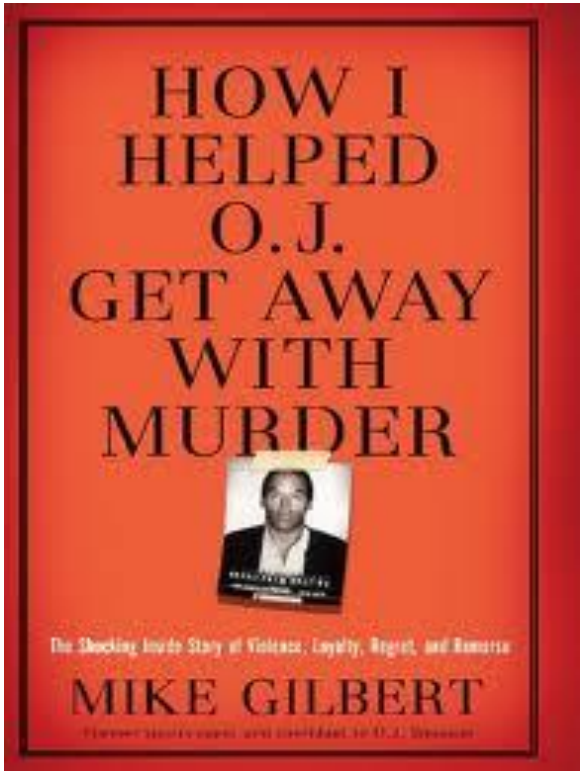
- Simpson was acquitted of murder charges and cannot be tried for the murders again in a criminal court.
- In the civil trial, the standard of proof is lower.
  - In the murder trial, the state had to prove Simpson committed the murders **beyond a reasonable doubt**, meaning that jurors had to be all but positive Simpson committed the murders to convict him.
  - In a civil trial, jury may decide for the plaintiffs if they determine that there is **at least a 50.1 percent probability** that Simpson is responsible.

[http://articles.cnn.com/1996-09-16/us/9609_16_simpson.case_1_murder-trial-sharon-rufo-ronald-goldman?_s=PM:US]

Related Books







- By O. J. Simpson .
- He puts forth a "hypothetical" description of the murders.
- Withdrawn by the publisher just before its release.
- In August 2007, the Goldman family was awarded the rights to the book to partially satisfy the civil judgment (in which O. J. Simpson. was found financially liable.)

[http://en.wikipedia.org/wiki/If_I_Did_It]